<u>Heat equation</u> (Diffusion equation).

$$c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right) + Q$$

K_0 thermal conductivity
ρ mass density
c specific heat

- Q source of heat energy.
- $u(x,t)$ temperature at x and t.
- $c, \rho\ K_0$ thermal coefficients (They may depend on x).

In the case of uniform rod (wire) thermal coefficients are constants. Then the heat equation becomes

$$c\rho \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q$$

If in addition $Q = 0$ (no source). then

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \qquad k = \frac{K_0}{c\rho}$$

k is called "thermal diffusivity"

Initial condition

$$u(x,0) = f(x)$$

Boundary condition

$$u(0,t), \ u(L,t)$$

without these values we can not predict $u(x,t)$.

or $$u(0,t) = u_B(t) \quad \text{prescribed temperature}$$

$$-K_0(0)\frac{\partial u}{\partial x}(0,t) = \phi(t) \quad \text{prescribed heat flow}$$

$$-K_0(0)\frac{\partial u}{\partial x}(0,t) = -H\left(u(0,t) - u_B(t)\right) \quad \text{Newton's law of cooling}$$

Heat equation in two and three dimensions

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$

$$\nabla^2 = \nabla\cdot\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

<u>steady state</u> : If the boundary conditions and any sources of thermal energy are independent of time, it is possible that there exist steady-state solutions to the heat equation, in this case

$$\nabla^2 u = 0 \quad \text{"Laplace equation"}$$

(if $Q \neq 0$, $\nabla^2 u = -Q/K_0$ Poisson eqn)

Applied Partial Differential Equations:
with Fourier Series and Boundary Value Problems
by Richard Haberman

Chapter 2 Method of Seperation of Variables

. The heat equation in one dimension.

$$u_t = k\, u_{xx} + \frac{\varphi(x,t)}{c\rho} \qquad \begin{array}{l} t>0 \\ 0<x<L \end{array}$$

. Initial condition

$$u(x,0) = f(x) \qquad 0<x<L$$

. Boundary conditions. As an example, if both ends of the rod have prescribed temperature then

$$u(0,t) = T_1(t) \qquad t>0$$
$$u(L,t) = T_2(t)$$

~~The are~~ Here $f(x)$, $T_1(t)$, $T_2(t)$ are given functions.

The method of separation of variables is used when the PDE and the BCs are linear and homogeneous

- <u>Linearity</u>. Let L be a differential operator. It is called a linear operator if

$$L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2)$$

for all constants c_1 and c_2 and for all functions u_1 and u_2.

The operator

$$\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}, \quad k > 0 \text{ a constant}$$

is a "linear operator"

$$\left(\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}\right)(c_1 u_1 + c_2 u_2)$$

$$= c_1\left(\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}\right)u_1 + c_2\left(\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}\right)u_2$$

- A <u>linear equation</u> is of the form

$$L(u) = f$$

where L is a linear operator and f is a known function. Examples of linear equations

i) $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2} + f(x,t)$ "Inhomogeneous heat equation"

ii) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ "Laplace equation"

iii) $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2} + \alpha(x,t)\, u$

Examples of equations which are not linear

i) $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2} + \alpha(x,t)\, u^2$ "nonlinear heat equation"

ii) $\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = \dfrac{\partial^3 u}{\partial x^3}$ "KdV equation"

- Linear homogeneous equation: let L be a linear operator and u is a function then

$$L(u) = 0$$

is a linear homogeneous equation. As an example

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0$$

is a linear homogeneous PDE known as the homogeneous heat (or diffusion) equation.

- Principle of superposition. Let L be a linear operator and $L(u)=0$ be a linear homogeneous equation. Let u_1 and u_2 be two functions solving $L(u)=0$ and c_1 and c_2 be any two arbitrary constants then $c_1 u_1 + c_2 u_2$ solves also the the homogeneous equation $L(c_1 u_1 + c_2 u_2)=0$

proof is simple: from the linearity of L we get

$$L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2) = 0$$

- The concepts of linearity and homogeneity also apply to boundary conditions. Examples of linear BCs

i) $u(0,t) = f(t)$

ii) $\frac{\partial u}{\partial x}(L,t) = g(t)$

iii) $\frac{\partial u}{\partial x}(0,t) = 0$

iv) $-K_0 \frac{\partial u}{\partial x}(L,t) = h\,[u(L,t) - g(t)]$

Examples of nonlinear BCs

i) $\frac{\partial u}{\partial x}(L,t) = u^2(L,t)$

ii) $u(0,t) = u(L,t)\,u(0,t) + \alpha.$

2.3 Heat equation with zero Temperatures at finite ends.

problem

H.Eq. $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$, $t>0,\ 0<x<L$

B.C. $u(0,t) = 0$, $t>0$
$u(L,t) = 0$,

I.C. $u(x,0) = f(x)$, $0<x<L$

Both ends immersed in a $0°$ temperature bath

. A constraint on $f(x)$ is $\underline{f(0)=0}$, $\underline{f(L)=0}$

- We are interested in predicting how the Initial thermal energy changes, $u(x,t)$
- In order to solve the inhomogeneous equation

$$u_t - k u_{xx} = Q(x,t)$$

we will need to know how to solve the homogeneous problem given above.

Solving the problem by Separation of Variables

i) Let

$$u(x,t) = \phi(x)\, G(t).$$

where ϕ is only a function of x and $G(t)$ only a function of t. The heat equation reduces to

$$\frac{1}{G}\frac{dG}{dt} = k\,\frac{1}{\phi}\frac{d^2\phi}{dx^2}$$

Left hand side is function of t only and the right hand side is function of x only. This is the consequence of separation of variables.

ii) Since both sides are different functions they can only be constants, the same constant, then

$$\frac{1}{kG}\frac{dG}{dt} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda.$$

where λ is an arbitrary constant, known as the "seperation constant". The minus sign is for convinience. Then the above equation means

$$\frac{dG}{dt} = -\lambda k G, \qquad \frac{d^2\phi}{dx^2} = -\lambda\phi$$

iii) BCs. The product $G(t)\,\phi(x) = u(x,t)$ must also satisfy the two homogeneous boundary conditions

$$u(0,t) = \phi(0)\,G(t) = 0$$

$$u(L,t) = \phi(L)\,G(t) = 0$$

The only possibility is $\phi(0) = \phi(L) = 0$

Rem: $G(t)$ can not be taken to be zero. If it is the case then $u(x,t) = 0$ for all x and t, but this is the trivial solution. We look for nontrivial solutions

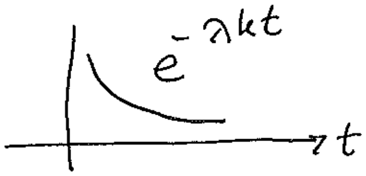
iv) Time dependent solution

$$\frac{dG}{dt} = -\lambda k G \ , \quad k > 0$$

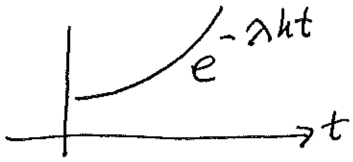
Solution: $G(t) = c\, e^{-\lambda k t}$, c is an arbitrary constant

Here the separation constant λ is arbitrary.

- If $\lambda > 0$, the ~~exponential function~~ the solution exponentially decays as t increases



- If $\lambda < 0$, the solution exponentially increases



. If $\lambda = 0$ the solution remains constant in time

Since this is a heat conduction problem and the temperature $u(x,t)$ is proportional $G(t)$ we do not expect the solution to grow exponentially in time. Thus we expect

$$\lambda \geq 0$$

v) Boundary value problem:

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \quad , \quad 0 < x < L$$

$$\phi(0) = 0$$

$$\phi(L) = 0$$

trivial solution $\phi(x) = 0 \quad \forall\, 0 \leq x \leq L$

The solution of the above linear homogeneous BVP is not unique. There is also a nontrivial solution. There do not exist nontrivial solutions for all values of the separation constant λ.

The BVP above is also called an eigenvalue problem

λ - eigenvalues

ϕ - Eigen function (vector)

of the linear operator $L = \dfrac{d^2}{dx^2}$

1) $\frac{d}{dx}[\phi_x^2 + \lambda \phi^2] = 2\phi_x(\phi_{xx} + \lambda\phi) = 0$

integrating in $[0,L]$ we get

$$\int_0^L \frac{d}{dx}(\phi_x^2 + \lambda\phi^2)\,dx = \phi_x^2 + \lambda\phi^2\Big|_0^L$$

$$\phi_x^2(L) = \phi_x^2(0) \quad , \quad \phi(L) = \phi(0) = 0$$

2) $$\int_0^x \frac{d}{dx}(\phi_x^2 + \lambda\phi^2)\,dx = \phi_x^2 + \lambda\phi^2 - [\phi_x^2(0) + \lambda\phi^2(0)] = 0$$

$$\phi_x^2 + \lambda\phi^2 = \phi_x^2(0) \quad , \quad \phi(0) = 0$$

for $\lambda \geq 0$ this always valid.

3) $$\int_x^L \frac{d}{dx}(\phi_x^2 + \lambda\phi^2)\,dx = \phi_x^2(L)^2 - (\phi_x^2 + \lambda\phi^2) = 0$$

$$\Rightarrow \quad \phi_x^2(L) = \phi_x^2 + \lambda\phi^2$$

again $\lambda \geq 0$

let us observe this fact by solving the boundary value problem

4) $$\phi_{xx} + \lambda \phi = 0 \qquad 0 < x < L$$

$$\phi(0) = \phi(L) = 0$$

$$\phi \phi_{xx} + \lambda \phi^2 = 0$$

$$\frac{d}{dx}(\phi \phi_x) - \phi_x^2 + \lambda \phi^2 = 0$$

$$\int_0^L \left[\frac{d}{dx}(\phi \phi_x) - \phi_x^2 + \lambda \phi^2 \right] dx = 0$$

$$\phi \phi_x \Big|_0^L + \int_0^L (-\phi_x^2 + \lambda \phi^2) dx = 0$$

$$\phi(L)\phi_x(L) - \phi(0)\phi_x(0) + \int_0^L (-\phi_x^2 + \lambda \phi^2) dx = 0$$

$$\Rightarrow \int_0^L (-\phi_x^2 + \lambda \phi^2) dx = 0$$

This is possible for $\lambda > 0$.

if $\lambda < 0$ then

$$\int_0^L (\phi_x^2 + |\lambda| \phi^2) dx = 0$$

$\Rightarrow \phi_x = 0$, $\phi(x) = 0 \quad \forall x \in (0, L)$. TRIVIAL SOLN.

if $\lambda = 0 \quad \phi_x(x) = 0 \quad \forall x \in (0, L)$

$\Rightarrow \phi(x) = $ constant but this constant should be zero due to the BCs TRIVIAL SOLN

- for $\lambda > 0$. Solution

$$\phi(x) = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x$$

where c_1 and c_2 are arbitrary constants.

BCs. $\phi(0) = 0$ implies $c_1 = 0$ and
$\phi(L) = 0$ implies $c_2 \sin\sqrt{\lambda}\, L = 0$

c_2 can not be zero (given trivial solution)

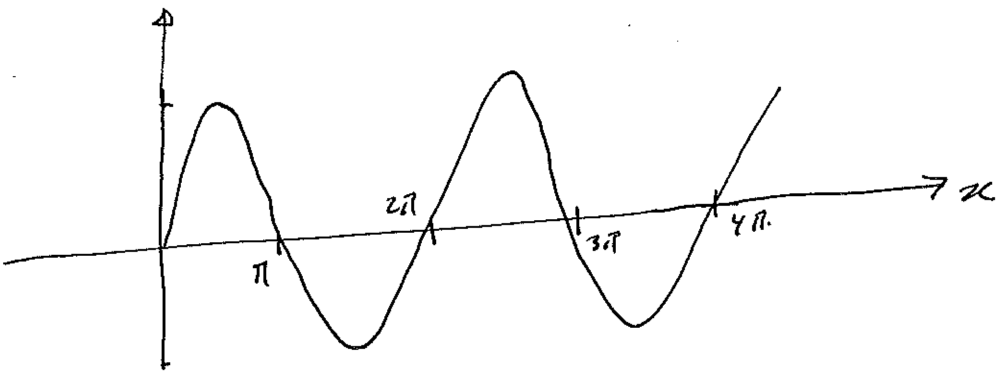
Hence

$$\sqrt{\lambda}\, L = n\pi, \quad n = 1, 2, 3, \dots \text{ positive integers.}$$

$$\Rightarrow \quad \lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, 3, \dots$$

Then eigen functions

$$\phi_n = C_n \sin\frac{n\pi}{L} x, \quad n = 1, 2, 3, \dots$$



- for $\lambda = 0$. Solution

$$\phi(x) = c_1 x + c_2$$

where c_1 and c_2 are arbitrary constants

BCs. $\phi(0) = 0$ implies $c_2 = 0$

$\phi(L) = 0$ implies $c_1 = 0$

$$\phi(x) = 0 \qquad \forall x \in [0, L]$$

This is the trivial solution. We say that $\lambda = 0$ is NOT an eigenvalue for this problem. It gives the trivial solution.

- for $\lambda < 0$. Solution (Let $\lambda = -s$)

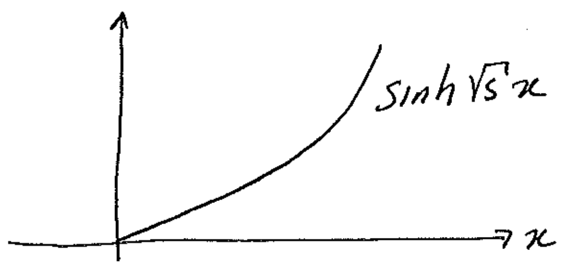
$$\phi(x) = c_1 \cosh \sqrt{s}\, x + c_2 \sinh \sqrt{s}\, x \quad , \; s > 0$$

where c_1 and c_2 are arbitrary constants.

BCs $\phi(0) = 0$ implies $c_1 = 0$

$\phi(L) = 0$ implies $c_2 \sinh \sqrt{s} L = 0$

The only solution is $c_2 = 0$, the trivial solution ($s > 0$).



So, the only solution for $\lambda < 0$ is the trivial solution which is not expected.

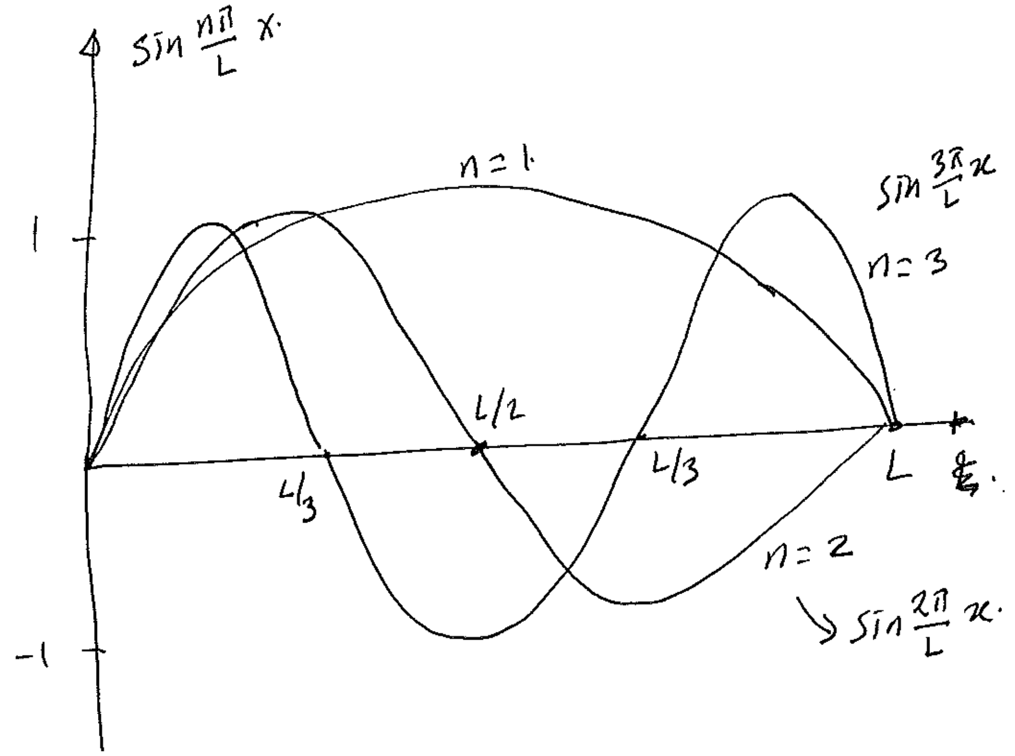
$\lambda < 0$ case was not a physical solution. In which case the time dependent part $G(t)$ grows exponentially.

As a summary. Solution of the eigenvalue problem.

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 , \quad n = 1,2,3,\dots$$

$$\phi_n(x) = C_n \sin \frac{n\pi}{L} x , \quad n = 1,2,3,\dots$$

solves the DE and the BCs.



vi) Product Solution: and the principle of Superposition. Since $u = G(t)\,\phi(x)$ we get

$$u_n(x,t) = B_n \sin\left(\frac{n\pi}{L}\right)x \; e^{-k\left(\frac{n\pi}{L}\right)^2 t}, \quad n = 1, 2, \ldots$$

for all values of $n = 1, 2, \ldots$ we have solutions of the same problem.

Solving the heat eqn. and BCs.

vii) A BV problem

PDE $\quad u_t = k\, u_{xx}, \quad t > 0, \quad x \in (0, L)$

BCs $\quad u(0,t) = 0 \qquad t > 0$
$\quad\quad\;\; u(L,t) = 0$

IC $\quad u(x,0) = 4 \sin \dfrac{3\pi x}{L}$.

Solution: Corresponds to $n = 3$.

$$u(x,t) = 4 \sin \frac{3\pi x}{L} \, e^{-k\frac{9\pi^2}{L^2} t}$$

viii) Another BV problem.

PDE $\quad u_t = k\, u_{xx}, \quad t > 0, \quad x \in (0, L)$

BCs $\quad u(0,t) = 0 \qquad t > 0$
$\quad\quad\;\; u(L,t) = 0$

IC $\quad u(x,0) = 4 \sin \frac{3\pi}{L} x + 7 \sin \frac{8\pi x}{L}$

Superposition of $n=3$ and $n=8$ solutions

$$u(x,t) = \underbrace{4\sin\frac{3\pi}{L}x\, e^{-k\left(\frac{3\pi}{L}\right)^2 t}}_{\text{solution for } n=3} + \underbrace{7\sin\frac{8\pi}{L}x\, e^{-k\left(\frac{8\pi}{L}\right)^2 t}}_{\text{solution for } n=8}$$

ix) For any IC. we have the superposition

$$u(x,t) = \sum_{n=1}^{M} B_n \sin\frac{n\pi}{L}x\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

for any M. at $t=0$

$$u(x,0) = f(x) = \sum_{n=1}^{M} B_n \sin\frac{n\pi}{L}x.$$

If the initial condition, the function $f(x)$, equals a finite sum of the appropriate sine functions, $\sin\frac{n\pi}{L}x$, then identifying the coefficients B_n we find the solution of the complete initial and boundary value problem. As an example

$$f(x) = 4\sin\frac{3\pi}{L}x + 7\sin\frac{8\pi}{L}x + 37\sin\frac{17\pi}{L}x.$$

$\Rightarrow$ $B_3 = 4$, $B_8 = 7$, $B_{17} = 37$ all others are zero

x) In general $f(x)$ is not a finite linear combination of the appropriate sine functions.

a) Any function $f(x)$ (with certain reasonable restrictions) can be approximated by a linear combination of $\sin \frac{n\pi}{L} x$

b) The approximation may not be very ~~smooth~~ good for small M, but gets to be a better and better approximation as M is increased

c) If we consider the limit as $M \to \infty$ then not only

$$f(x) = \sum_{n=1}^{M}{}' B_n \sin \frac{n\pi}{L} x \qquad \text{as } M \to \infty$$

is the best approximation to $f(x)$ using the combinations of the eigen functions, but the resulting infinite series converge to $f(x)$. (with some restrictions on $f(x)$ to be discussed) Hence for any initial condition $f(x)$ can be written as a Fourier series

$$f(x) = \sum_{n=1}^{\infty}{}' B_n \sin \frac{n\pi}{L} x.$$

and the infinite series

$$u(x,t) = \sum_{n=1}^{\infty}{}' B_n \sin \frac{n\pi}{L} x \, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

solves the heat conduction problem we discussed.

xi) How to find B_n's. We use the orthogonality of the eigenfunctions. $\sin\frac{n\pi}{L}x$. in $[0,L]$

$$\int_0^L \sin\frac{n\pi}{L}x \sin\frac{m\pi}{L}x \, dx = \begin{cases} 0 & n\neq m \\ L/2 & n=m \end{cases}$$

using this property we get

$$\int_0^L f(x) \sin\frac{m\pi}{L}x\, dx = \sum{}' B_n \int_0^L \sin\frac{n\pi}{L}x \sin\frac{m\pi}{L} dx$$

$$= \frac{L}{2} B_m.$$

$$B_m = \frac{2}{L}\int_0^L f(x) \sin\frac{m\pi}{L}x \, dx \quad , \; n=1,2,\ldots$$

Then the complete solution of the heat conduction problem is

$$u(x,t) = \sum_{n=1}^{\infty}{}' B_n \sin\frac{n\pi}{L}x \, e^{-k\left(\frac{n\pi}{L}\right)^2 t} \quad , \; t>0$$

and B_n's are found from the above equation

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L}x \, dx$$

Example:

PDE $\quad u_t = k\, u_{xx} \qquad t>0,\ x \in (0,L)$

BC $\quad \begin{array}{l} u(0,t)=0 \\ u(L,t)=0 \end{array} \quad t>0$

IC $\quad u(x,0) = u_0$ const. $,\ 0<x<L$

$$B_n = \frac{2}{L}\int_0^L u_0 \sin\frac{n\pi}{L}x\,dx = \frac{2u_0}{L}\int_0^L \sin\frac{n\pi}{L}x\,dx$$

$$= \frac{2u_0}{L}\left[-\frac{L}{n\pi}\cos\left(\frac{n\pi}{L}x\right)\Big|_0^L\right]$$

$$= \frac{2u_0}{n\pi}\left[-\cos(n\pi)+1\right] = \frac{2u_0}{n\pi}\left(1-(-1)^n\right)$$

$$= \begin{cases} 0 & n = \text{even} \\ \dfrac{4u_0}{n\pi} & n = \text{odd.} \end{cases}$$

$$u(x,t) = \sum_{n\ \text{odd}} \frac{4u_0}{n\pi} \sin\frac{n\pi}{L}x\ e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$\approx \frac{4u_0}{\pi}\sin\frac{\pi x}{L}\ e^{-k\left(\frac{\pi}{L}\right)^2 t} + \frac{2u_0}{\pi}\sin\frac{2\pi}{L}x\ e^{-k\left(\frac{2\pi}{L}\right)^2 t} + \dots$$

smaller than the first term.

xii) convergence problem: We have two infinite sum

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \lambda_n x \, e^{-k\lambda_n^2 t} \qquad t > 0$$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \lambda_n x \quad , \quad \lambda_n = \frac{n\pi}{L}, \; n = 1, 2, \ldots$$

- In order that $u(x,t)$ series to satisfy the heat equation, the infinite series should be term by term differentiable. To have this property (to be the solution) the RHS infinite sum should be uniformly convergent. For this purpose it is enough to have the initial value $f(x)$ be bounded in $[0, L]$

$$|f(x)| \leq M \quad \text{(positive constant)}.$$

since

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \lambda_n x \, dx$$

Then

$$|B_n| \leq \frac{2}{L} \int_0^L |f(x)| \, dx \leq \frac{2}{L} M \cdot L$$

$$\leq 2M \qquad \text{is also bounded}$$

The infinite series corresponding to u, u_x, u_{xx}, u_t must be uniformly continuous.

$$u_{xx} \Rightarrow -\sum_{n=1}^{\infty}{}' B_n \lambda_n^2 \sin\lambda_n x \, e^{-k\lambda_n^2 t}$$

$$\left| \sum_{n=1}^{\infty}{}' B_n \lambda_n^2 \sin\lambda_n x \, e^{-k\lambda_n^2 t} \right| \le \sum_{n=1}^{\infty}{}' |B_n| \lambda_n^2 e^{-k\lambda_n^2 t}$$

if $t > \tilde{t} > 0$ then

$$\left| \sum_{n=1}^{\infty}{}' B_n \lambda_n^2 \sin\lambda_n x \, e^{-k\lambda_n^2 t} \right| \le 2M \sum_{n=1}^{\infty}{}' \lambda_n^2 e^{-k\lambda_n^2 \tilde{t}}$$

The infinite series $\sum_{n=1}^{\infty}{}' \lambda_n^2 e^{-k\lambda_n^2 \tilde{t}}$ is uniformly conv. by Weirstrass test.

(*) Hence in order that the solution $u(x,t)$ to be the solution $f(x)$ must be bounded in $[0,L]$

The sum corresponding to $f(x)$ must be also uniformly convergent

$$\left| \sum{}' B_n \sin\lambda_n x \right| \le \sum_{n=1}^{\infty}{}' |B_n|$$

hence

$$\sum_{n=1}^{\infty}{}' |B_n| < \infty$$

(*) If $f(x)$ is bounded then $u(x,t)$ solves the DE

For this purpose we need to find conditions on $f(x)$.

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\lambda_n x\,dx \quad , \quad n=1,2,\dots$$

integration by part

$$B_n = \frac{2}{L}\left[-\frac{1}{\lambda_n}\cos\lambda_n x\; f(x)\Big|_0^L + \frac{1}{\lambda_n}\int_0^L f'(x)\cos\lambda_n x\,dx\right]$$

Let $f(0)=f(L)=0$, then

$$B_n = \frac{2}{L}\,\frac{1}{\lambda_n}\int_0^L f'(x)\cos\lambda_n x\,dx$$

$$= +\frac{2}{\lambda_n^2 L}\int_0^L f''\sin\lambda_n x\,dx$$

$$|B_n| \leq \frac{2}{\lambda_n^2 L}\int_0^L |f''|\,dx$$

Let $|f''|\leq M'$ in $[0,L]$, then $(M'>0)$

$$|B_n| \leq \frac{2}{\lambda_n^2}M'$$

Then $\sum |B_n| \leq 2M'\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2} = \frac{2M'}{\pi^2}L^2\sum_{n=1}^{\infty}\frac{1}{n^2}$

$$\leq \frac{2M'}{\pi^2}L^2\,\pi^2/6 \leq \frac{M'L^2}{3}$$

Hence if f has a continuous second derivative in $(0,L)$ and $f(0)=f(L)=0$ then the infinite sum for $f(x)$

$$\sum B_n \sin \lambda_n x$$

is uniformly convergent.

As a result we have the following thm.

<u>Theorem</u>: The heat conduction problem has a solution if $f(x)$ is bounded in $[0,L]$. Furthermore the initial condition is satisfied if $f(0)=f(L)=0$ and $f(x)$ has continuous second derivative in $(0,L)$.

<u>The eigenvalue problem:</u> let $L = \frac{d^2}{dx^2}$ be the differential operator. Let

$$L\,\phi_n = \lambda_n \phi_n \qquad n = 1,2,\ldots$$

λ_n eigenvalues of L

ϕ_n eigenvectors of L

Inner product. let f, g be function in $[0,L]$ then

$$\langle f \mid g \rangle = \int_0^L \bar{f}(x)\, g(x)\, dx$$

satisfies all properties of an inner product.

In addition to the eigenvalue problem we have also

$$\phi_n(0) = \phi_n(L) = 0 \qquad n = 1,2,\ldots$$

i) λ_n's are real

$$L\,\phi_n = \lambda_n \phi_n$$
$$L\,\bar{\phi}_n = \bar{\lambda}_n \bar{\phi}_n$$

$$\phi_n L \bar{\phi}_n = \bar{\lambda}_n \bar{\phi}_n \phi_n$$
$$\bar{\phi}\, L \phi_n = \lambda_n \phi_n \bar{\phi}_n$$

$$\phi_n L \bar{\phi}_n - \bar{\phi}_n L \phi_n = (\bar{\lambda}_n - \lambda_n)\, \phi_n \bar{\phi}_n$$

$$\int_0^L (\phi_n L \bar{\phi}_n - \bar{\phi}_n L \phi_n)\, dx = (\bar{\lambda}_n - \lambda_n) \int_0^L \phi_n \bar{\phi}_n$$

$$\int_0^L (\phi_n L\bar{\phi}_n - \bar{\phi}_n L\phi_n)\,dx = \int_0^L \left(\phi_n \frac{d^2\bar{\phi}_n}{dx^2} - \bar{\phi}_n \frac{d^2\phi_n}{dx^2}\right)dx$$

$$= \int_0^L \left[\frac{d}{dx}\left(\phi_n \frac{d\bar{\phi}_n}{dx}\right) - \cancel{\frac{d\phi_n}{dx}\frac{d\bar{\phi}_n}{dx}} - \frac{d}{dx}\left(\bar{\phi}_n \frac{d\phi_n}{dx}\right) + \cancel{\frac{d\phi_n}{dx}\frac{d\bar{\phi}_n}{dx}}\right]dx$$

$$= \left(\phi_n \frac{d\bar{\phi}_n}{dx} - \bar{\phi}_n \frac{d\phi_n}{dx}\right)\Bigg|_0^L = 0$$

hence

$$(\lambda_n - \bar{\lambda}_n)\int_0^L \phi_n \bar{\phi}_n\,dx = 0 \quad \Rightarrow \quad \bar{\lambda}_n = \lambda_n \qquad \forall\, n = 1, 2, \ldots$$

ii) Eigenfunctions are orthogonal. (

$$\bar{\phi}_m L\phi_n = \lambda_n \phi_n \bar{\phi}_m$$

$$\int_0^L \bar{\phi}_m L\phi_n\,dx = \lambda_n \int_0^L \phi_n \bar{\phi}_m\,dx$$

$$= -\int_0^L \frac{d\bar{\phi}_m}{dx}\frac{d\phi_n}{dx}\,dx = \int_0^L \phi_n \frac{d^2\bar{\phi}_m}{dx^2}\,dx$$

$$= \int_0^L \phi_n \bar{\lambda}_m \bar{\phi}_m\,dx \Rightarrow$$

$$(\bar{\lambda}_m - \lambda_n)\int_0^L \phi_n \bar{\phi}_m\,dx = 0$$

$$\Rightarrow \quad (\lambda_n - \lambda_m) \langle \phi_m, \phi_n \rangle = 0$$

for $\quad \lambda_n \neq \lambda_m$

$$\langle \phi_m, \phi_n \rangle = 0 \qquad \text{for } m \neq n.$$

hence for different eigenvalues the corresponding eigenfunctions are orthogonal

Heat Conduction in a Rod with Insulated Ends

PDE $\quad \dfrac{\partial u}{\partial t} = k \dfrac{\partial^2 u}{\partial x^2} \quad , \quad t > 0, \; 0 < x < L$

BCS $\quad \dfrac{\partial u}{\partial x}(0,t) = 0 \qquad t > 0$

$\quad \dfrac{\partial u}{\partial x}(L,t) = 0$

IC $\quad u(x,0) = f(x) \quad , \quad 0 < x < L$

This problem is quite similar to the previous problem with zero temperatures at the end points. The only difference is the BV problem.

$$\frac{d^2\phi}{dx^2} = -\lambda \phi$$

$$\frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(L) = 0$$

for $\lambda > 0$.

$$\phi(x) = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x.$$

$$\frac{d\phi}{dx} = -c_1\sqrt{\lambda} \sin\sqrt{\lambda}\, x + c_2\sqrt{\lambda} \cos\sqrt{\lambda}\, x$$

$$\frac{d\phi}{dx}(0) = 0 \implies c_2 = 0$$

$$\frac{d\phi}{dx}(L) = 0 \implies \sqrt{\lambda}\, L = n\pi \quad , \; n = 1, 2, \ldots,$$

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

$$\phi_n(x) = c_n \cos\frac{n\pi}{L} x \qquad n = 1, 2, \ldots$$

$\lambda = 0$

$$\phi(x) = c_1 + c_2 x$$

$$\frac{d\phi}{dx} = c_2 \quad \Rightarrow \quad \frac{d\phi}{dx}(0) = c_2 = 0$$

$$\Rightarrow \quad \frac{d\phi}{dx}(L) = 0.$$

$$\phi(x) = c_1.$$

superposition

$$\phi(x) = c_0 + \sum_{n=1}^{\infty} c_n \cos\frac{n\pi}{L} x.$$

There are no eigenfunctions for $\lambda < 0$ (Exercise)

Hence

$$u(x,t) = c_0 + \sum_{n=1}^{\infty} c_n \cos\frac{n\pi}{L} x \, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

c_0, c_n $(n = 1, 2, \ldots)$ are arbitrary constants.

Initial condition

$$u(x,0) = f(x) = c_0 + \sum_{n=1}^{\infty} c_n \cos\frac{n\pi}{L} x$$

Orthogonality of cosine

$$\int_0^L \cos\frac{m\pi}{L}x \cos\frac{n\pi}{L}x\,dx = \begin{cases} 0 & m\neq n \\ \frac{L}{2} & m=n\neq 0 \\ L & m=n=0 \end{cases}$$

$$\int_0^L 1\cdot\cos\frac{m\pi}{L}\,dx = 0 \quad \forall m \qquad \int_0^L 1\cdot 1\,dx = L$$

$\{1, \cos\frac{m\pi}{L}x\}$ orthogonal set

Then

$$c_0 = \frac{1}{L}\int_0^L f(x)\,dx$$ average value of the temperature

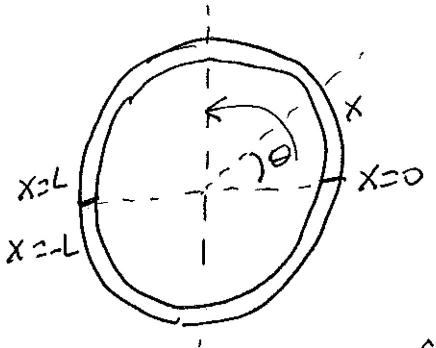
$$= \lim_{t\to\infty} u(x,t).$$

$$c_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx.$$

Existence of solution requires some constraints on the initial function $f(x)$. What are they?

Uniform convergence requires some restrictions on $f(x)$?

Heat Conduction in a thin Circular Ring



Length of the circle

$$2\pi r = 2L$$

$$r = \frac{L}{\pi}$$

$$r\theta = x. \quad \Rightarrow \theta = \frac{x}{r} = \frac{x}{L}\pi.$$

Very thin rod in circular shape.
same temperature everywhere in the cross section
One dimensional problem.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} = \frac{k}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$t > 0 \quad , -\pi < \theta < \pi$$

At the end point $\theta = \pm\pi$ u and $\frac{\partial u}{\partial \theta}$ are continuous.

BV problem.

$$\frac{d^2\phi}{d\theta^2} = -\lambda\phi \qquad -\pi < \theta < \pi$$

$$\phi(\pi) = \phi(-\pi)$$

$$\frac{d\phi}{d\theta}(\pi) = \frac{d\phi}{d\theta}(-\pi).$$

. for $\lambda > 0$.

$$\phi(\theta) = C_1 \cos\sqrt{\lambda}\,\theta + C_2 \sin\sqrt{\lambda}\,\theta.$$

i) $\phi(\pi) = \phi(-\pi)$

$$C_2 \sin\sqrt{\lambda}\,\pi = 0$$

ii) $\frac{d\phi}{d\theta}(\pi) = \frac{d\phi}{d\theta}(-\pi)$

$$-C_1\sqrt{\lambda}\,\sin\sqrt{\lambda}\,\pi = 0$$

the only nontrivial solution

$$\sqrt{\lambda}\,\pi = n\pi \quad , n = 1, 2, \dots$$

$$\lambda = n^2 \qquad n = 1, 2, \dots$$

Eigen values $\lambda = n^2$

Eigen functions $\cos n\theta, \ \sin n\theta.$

for $\lambda = 0$

$$\phi(\theta) = C_1 + C_2\theta$$

$\phi(\pi) = \phi(-\pi) \Rightarrow C_1 + C_2\pi = C_1 - C_2\pi \Rightarrow C_2 = 0$

$\phi'(\pi) = \phi'(-\pi) \qquad C_2 = C_2$

$$\phi(\theta) = C_1$$

Eigen value $\lambda = 0$

Eigen funct 1.

Super position

$$u(\theta,t) = C_0 + \sum_{n=1}^{\infty} C_n \cos n\theta \, e^{-k\frac{\pi^2 n^2}{L^2}t} + \sum_{n=1}^{\infty} B_n \sin n\theta \, e^{-k\frac{n^2\pi^2}{L^2}t}$$

where C_0, C_n, B_n $(n = 1, 2, \dots)$ are arbitrary constants

Initial condition

$$u(\theta,0) = f(\theta) = C_0 + \sum_{n=1}^{\infty} C_n \cos n\theta + \sum_{n=1}^{\infty} B_n \sin n\theta$$

Orthogonality

i) $$\int_{-L}^{L} \sin\frac{n\pi}{L}x \, \sin\frac{m\pi}{L}x \, dx = \begin{cases} 0 & m \neq n \\ \frac{L}{2} & m = n \end{cases}$$

ii) $$\int_{-L}^{L} \cos\frac{n\pi}{L}x \, \cos\frac{m\pi}{L} \, dx = \begin{cases} 0 & m \neq n \\ \frac{L}{2} & m = n \neq 0 \\ L & m = n = 0 \end{cases}$$

iii) $$\int_{-L}^{L} \cos\frac{n\pi}{L}x \, \sin\frac{m\pi}{L} \, dx = 0 \quad \forall\, m, n \geq 1$$

iv) $$\int_{0}^{L}$$

Orthogonality: $-\pi \leq \theta \leq \pi$

i) $$\int_{-\pi}^{\pi} \sin n\theta \cos m\theta \, d\theta = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$$

ii) $$\int_{-\pi}^{\pi} \cos n\theta \cos m\theta \, d\theta = \begin{cases} 0 & m \neq n \\ \pi & m = n \neq 0 \\ 2\pi & m = n = 0 \end{cases}$$

iii) $$\int_{-\pi}^{\pi} \sin n\theta \cos m\theta \, d\theta = 0 \qquad \forall m, n$$

iv) $$\int_{-\pi}^{\pi} 1 . \sin n\theta \, d\theta = 0 \qquad \forall n$$

v) $$\int_{-\pi}^{\pi} 1 . \cos n\theta \, d\theta = 0 \qquad \forall n$$

vi $$\int_{-\pi}^{\pi} 1 . 1 \, d\theta = 2\pi$$

$$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) d\theta, \quad C_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta .$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta \qquad n = 1, 2, 3, \dots,$$

mixed boundary boundary conditions

$u_t = k\, u_{xx} \qquad t > 0,\ 0 < x < L$

$u(0,t) = 0$

$u_x(L,t) = 0$

$u(x,0) = f(x)$

$\phi_{xx} + \lambda \phi = 0$

$\phi(0) = 0$

$\phi'(L) = 0$

$\lambda > 0$

$\phi = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x$

$\phi(0) = c_1 = 0$

$\phi'(L) = c_2 \sqrt{\lambda} \cos\sqrt{\lambda}\, L = 0$

$\sqrt{\lambda}\, L = \left(n + \frac{1}{2}\right)\pi \qquad n = 0, 1, 2, \ldots$

$\lambda_n = \left(\frac{(2n+1)\pi}{2L}\right)^2 \qquad n = 0, 1, 2, \ldots$

$\phi_n = c_n \sin\left(\sqrt{\lambda_n}\, x\right)$

$\lambda = 0$

$\phi = c_1 + c_2 x$

$\phi(0) = c_1 = 0$

$\phi'(L) = c_2 L = 0 \quad c_2 = 0$ (trivial)

$\lambda = 0$ is NOT an eigenvalue

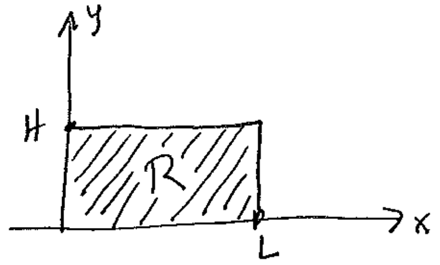
$\lambda < 0$

is NOT also an eigenvalue

2.5 Laplace's Equation.

Laplace's Equation Inside a Rectangle

$$R = \{ (x,y) \in \mathbb{R}^2 \mid 0 \le x \le L, 0 \le y \le H \}$$



$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{inside } R$$

BCs: $u(0,y) = g_1(y)$, $u(L,y) = g_2(y)$, $0 < x < L$

$u(x,0) = f_1(x)$, $u(x,H) = f_2(x)$, $0 < y < H$

where $f_1(x), f_2(x), g_1(y)$, and $g_2(y)$ are given functions of x and y, respectively.

- PDE is linear and homogeneous, BCs are also linear but not homogeneous, hence the method of separation of variables is NOT suitable to solve the boundary value problem.
- We can use the principle of superposition to solve this boundary value problem. Let

$$u = u_1 + u_2 + u_3 + u_4$$

where u_i is the solution of the boundary value problem with one nonhomogeneous boundary condition but the other three are homogeneous.

1) $\nabla^2 u_1 = 0$ inside of R.

$u_1(0, y) = g_1(y)$

$u_1(L, y) = 0$

$u_1(x, 0) = 0$

$u_2(x, H) = 0$

2) $\nabla^2 u_2 = 0$ inside of R.

$u_2(0, y) = 0$

$u_2(L, y) = g_2(y)$

$u_1(x, 0) = 0$

$u_2(x, H) = 0$

3) $\nabla^2 u_3 = 0$ inside of R

$u_3(0, y) = 0$

$u_3(L, y) = 0$

$v_3(x, 0) = f_1(x)$

$u_3(x, H) = 0$

4) $\nabla^2 u_4 = 0$ inside of R

$u_4(0, y) = 0$

$u_4(L, y) = 0$

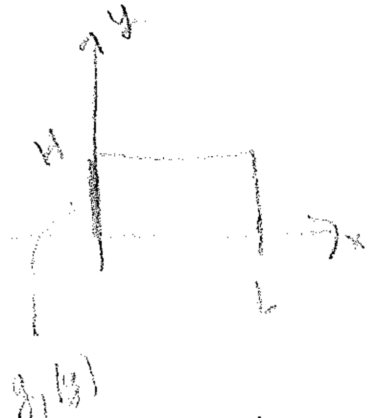
$u_4(x, 0) = 0$

$u_4(x, L) = f_2(x)$.

Hence if $u = u_1 + u_2 + u_3 + u_4$ in R, then $\nabla^2 u = 0$ inside of R and satisfies inhomogeneous conditions.

If we solve one of these BV problem all the other are similar

PDE $$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0 \quad \text{in } R$$



$$u_1(0, y) = g_1(y), \quad 0 < y < L$$
$$u_1(L, y) = 0$$
$$u_1(x, 0) = 0, \quad 0 < x < L$$
$$u_1(x, H) = 0$$

solution: by the method of separation of variables

i) $u(x,y) = h(x)\,\phi(y)$.

$$\frac{h''}{h} + \frac{\phi''}{\phi} = 0 \Rightarrow \frac{\phi''}{\phi} = -\lambda, \quad \frac{h''}{h} = \lambda$$

where λ is the separation constant. Hence we have

$$\phi'' + \lambda\phi = 0 \quad \text{with} \quad \phi(0) = 0, \ \phi(H) = 0$$

and

$$h'' = \lambda h \quad \text{with} \quad h(L) = 0$$

ii) Solutions of ϕ. $\lambda > 0$

$$\phi(y) = c_1 \cos\sqrt{\lambda}\, y + c_2 \sin\sqrt{\lambda}\, y$$
$$\phi(0) = 0 \Rightarrow c_1 = 0$$
$$\phi(H) = 0 \Rightarrow \lambda \Rightarrow \lambda_n = \left(\frac{n\pi}{H}\right)^2 \quad n = 1, 2, \ldots$$

$$\lambda_n = \left(\frac{n\pi}{H}\right)^2$$

$$\phi_n = C_n \sin\left(\frac{n\pi}{H} y\right) \qquad n = 1, 2, \ldots$$

$\lambda = 0$ $\qquad \phi = C_1 + C_2 y$

$$\phi(0) = C_1 = 0$$

$$\phi(H) = C_2 H = 0 \Rightarrow C_2 = 0 \quad \text{trivial soln.}$$

hence $\lambda = 0$ is NOT an eigenvalue.

$\lambda < 0$ No solution (nontrivial) exist. Hence there are NO eigenvalues with $\lambda < 0$

iii) solutions of h. We have only $\lambda > 0$

$$h(x) = B_1 \sinh\sqrt{\lambda}\, x + B_2 \cosh\sqrt{\lambda}\, x$$

$$h(L) = B_1 \sinh\sqrt{\lambda}\, L + B_2 \cosh\sqrt{\lambda}\, L = 0$$

$$\Rightarrow h(x) = B_1 \sinh\sqrt{\lambda}\, x - B_1 \frac{\sinh\sqrt{\lambda}\, L}{\cosh\sqrt{\lambda}\, L} \cosh\sqrt{\lambda}\, x$$

$$= \frac{B_1}{\cosh\sqrt{\lambda}\, L} \sinh\sqrt{\lambda}\,(x - L)$$

$\cosh\sqrt{\lambda}\, L \geq 1.$ $\qquad h(x) = \bar{B}_1 \sinh\frac{n\pi}{H}(x-L)$

$$n = 1, 2, \ldots$$

iv) $u_n(x,y) = h_n(x)\, \phi_n(y) = A_n \sin\left(\frac{n\pi}{H} y\right) \sinh\frac{n\pi}{H}(x-L)$

$$n = 1, 2, \ldots$$

Superposing all solutions

$$u_1(x,y) = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi}{H}(x-L) \sin\left(\frac{n\pi}{H} y\right)$$

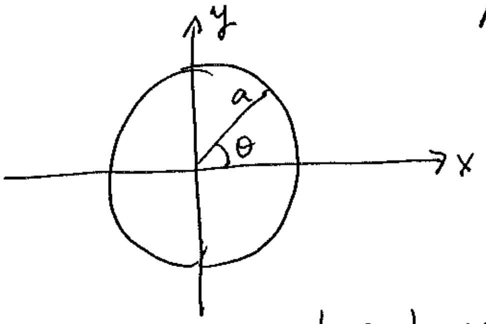
v) BC at $y=0$

$$u_1(x,0) = g_1(y) = -\sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{H}L\right) \sin\left(\frac{n\pi}{H} y\right)$$

$$A_n = -\frac{2}{H \sinh\left(\frac{n\pi}{H}L\right)} \int_0^H g_1(y) \sin\left(\frac{n\pi}{H} y\right) dy \qquad n=1,2,\ldots$$

Here $\sinh\left(\frac{n\pi}{H}L\right)$ is never zero

Laplace's equation for a Circular Disk



Any point inside the disk (r, θ). with

$$0 \le r < a \quad , \quad 0 \le \theta \le 2\pi$$

boundary of the disk $r = a$.

Laplace equation $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

in Cartesian coordinates.

$$x = r\cos\theta \ , \quad y = r\sin\theta$$

$$r = \sqrt{x^2+y^2} \quad , \quad \tan\theta = \frac{x}{y}$$

$$u_x = u_r \frac{x}{r} + u_\theta \frac{1}{r} \frac{\cos^2\theta}{\sin\theta}$$

- - -

In polar coordinates

1) PDE $\quad \nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad$ in D.

2) BC : $\quad u(a, \theta) = f(\theta) \qquad 0 < \theta < 2\pi$

3) periodicity

$$u(r, -\pi) = u(r, \pi) \qquad \text{in Disk.}$$

$$u_\theta(r, -\pi) = u_\theta(r, \pi)$$

equivalent to $\quad u(r, \theta + 2\pi) = u(r, \theta)$

4). Boundedness condition at the origin

$$|u(0, \theta)| < \infty$$

<u>Solution</u>: By the method of separation of variables

i) $u(r,\theta) = \phi(\theta)\, G(r)$.

$$\frac{1}{Gr}\frac{d}{dr}\left(r\frac{dG}{dr}\right) + \frac{1}{\phi r^2}\frac{d^2\phi}{d\theta^2} = 0$$

or
$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2} = \lambda$$

where λ is the separation constant. Homogeneous Boundary conditions

$$\frac{d^2\phi}{d\theta^2} + \lambda\phi = 0$$

with
$$\phi(\pi) = \phi(-\pi).$$
$$\phi'(\pi) = \phi'(-\pi).$$

and
$$r\frac{d}{dr}\left(r\frac{dG}{dr}\right) - \lambda G = 0$$

ii) Solution of the BV problem.

$$\phi'' + \lambda\phi = 0 \;, \quad \phi(\pi) = \phi(-\pi) \;, \quad \phi'(\pi) = \phi'(-\pi)$$

we solved this problem

eigen values

$$\lambda_n = n^2 \qquad n = 0, 1, 2, \ldots$$

eigenvectors are $1, \ \sin n\theta, \ \cos n\theta$

(iii) Solution of the radial equation

$$r \frac{d}{dr}\left(r \frac{dG}{dr}\right) = n^2 G$$

$$r^2 G'' + r G' - n^2 G = 0$$

$$r = e^y \qquad G' = G_y \frac{1}{r} \qquad G'' = -\frac{1}{r^2} G_y + \frac{1}{r^2} G_{yy}$$

$$G_{yy} - G_y + G_y - n^2 G = 0 \quad \Rightarrow \quad G_{yy} - n^2 G = 0$$

$$G = a e^{ny} + b e^{-ny} = a_n r^n + b_n r^{-n} \qquad , n = 1, 2, \ldots$$

$$= a_1 + b_1 y = a' + b_1' \ln r \qquad n = 0$$

boundedness imply $b_n = 0, \ b' = 0 \Rightarrow$

$$G = a_n r^n, \qquad n = 1, 2, \ldots$$

hence

$$u = A_0 + A_n \cos n\theta \, r^n + B_n \sin n\theta \, r^n$$

superposition

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) r^n$$

iv) Boundary condition

$$u(a,\theta) = f(\theta) = A_0 + \sum_{n=1}^{\infty} a^n \left(A_n \cos n\theta + B_n \sin n\theta \right)$$

using the orthogonality of the set $1, \cos n\theta, \sin n\theta$ in $(-\pi, \pi)$ we get

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta$$

$$A_n = \frac{1}{\pi a^n} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta \qquad n = 1, 2, ..$$

$$B_n = \frac{1}{\pi a^n} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta \qquad n = 1, 2, .$$

It is interesting that

$$A_0 = Av(f)$$

and

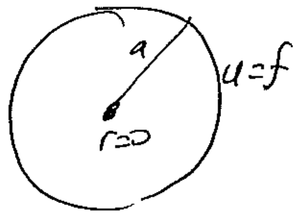
$$u(0,\theta) = A_0 = Av(f)$$

2.5.4. Qualitative Properties of Laplace's Equation

i) Mean Value theorem

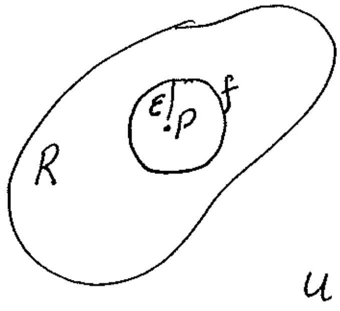
Solution of Laplace's equation in a disk of radius a. Temperature at the origin, $r=0$ is

$$u(0,\theta) = A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$



$$= av(f)$$

This is the mean value property of Laplace's equation. This property holds in general. Suppose that we wish to solve Laplace's equation in any region R. Consider any point $P \in R$ and a circle of any radius ε (such that circle is inside R). let the temperature on the circle be $f(\theta)$. using polar coordinates at P.



The temperature at P is the average of the temperature along any circle of radius ε centered at P.

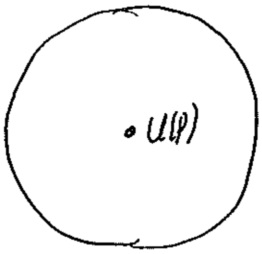
$$u(P) = av(f)$$

ii) <u>Maximum Principles</u>. We can use the mean value property to prove the maximum principle for the Laplace's equation.

In steady state (no t - dependence). The temperature ~~attain its m~~ cannot attain its maximum in the interior (unless the temperature is a constant everywhere)

The proof is by contradiction. Suppose that the maximum is at point P. However this should be the average of temeperatures at points of circle of radius ε

$u(x,y) < u(P) \qquad \forall (x,y \in R)$



It is impossible for the temperature at P to be larger. At some points of the circle temperature must be larger than $u(P)$ in order that $av(f)$ to be $u(P)$.

But this contradicts the assumptions, which thus can not hold.

By letting $\psi = -u$ we can also show that the temperature cannot attain its minimum in the interior.

As a summary in steady state (Laplace's equation) the maximum and minimum temperatures occur on the boundary.

iii) Wellposedness and uniqueness.

A well posed problem (an initial and boundary value problem).

a) There exists a solution

b) the solution is unique

c) solution continuously depends on the data (solution is stable).

- solution varies a small amount if the data are slightly changed.

A boundary value problem

$$\nabla^2 \psi = 0 \quad \text{in } R$$

$$\psi\big|_{\partial R} = f$$

This is a well posed problem. Stability. Let ψ_1 and ψ_2 be solutions corresponding to boundary value f_1 and f_2 respectively. Then $\psi = \psi_1 - \psi_2$

$$\nabla^2 \psi = 0 \quad \text{in } R \quad \text{and} \quad \psi\big|_{\partial R} = f_1 - f_2$$

since by use of the maximum and minimum principle

$$\min(f_1 - f_2) < \psi < \max(f_1 - f_2)$$

If $|f_1 - f_2| < \varepsilon \Rightarrow -\varepsilon < \psi < \varepsilon$

or $|\psi| < \varepsilon$ which means stability

iv) Solvability condition

$$\nabla^2 \psi = 0 \quad \text{in } R$$

$$\iint_R \nabla^2 \psi \, dx\, dy = \iint_R \nabla \cdot (\nabla \psi) \, dx\, dy$$

$$= \oint_{\partial R} \hat{n} \cdot \vec{\nabla} \psi \, ds = 0.$$

if the BC is the specification of

$$\hat{n} \cdot \vec{\nabla} \psi \equiv \frac{\partial \psi}{\partial n} = f \qquad \text{on } \partial R.$$

$\Rightarrow$ Solvability condition

$$\oint_{\partial R} f \, ds = 0$$

If f doesnot satisfy this condition there exist no solution of the BV problem.

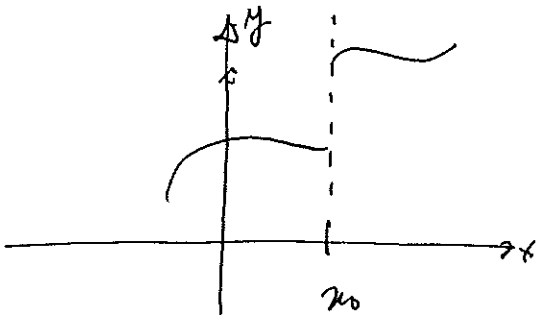
3. Fourier Series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

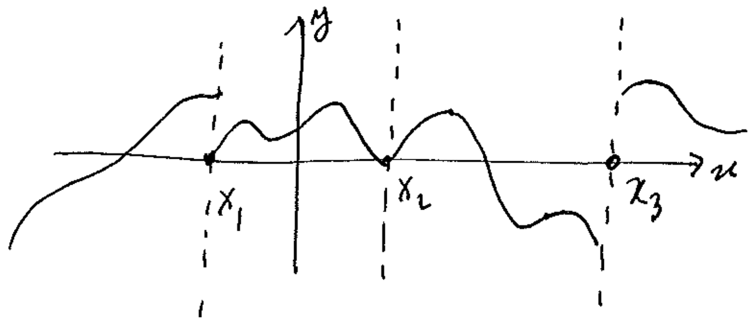
- Does the infinite series converge?
- Does it converge to $f(x)$?
- Is the resulting infinite series really a solution of the partial differential equation?

Some definitions

i) A function $f(x)$ is ~~slid~~ piecewise smooth (on some interval) if the interval ~~is~~ can be broken up into pieces such that in each piece the function is continuous and its first derivative $\frac{df}{dx}$ is also continuous



jump discontinuity



a piecewise smooth function.

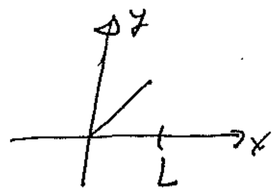
ii) A function $f(x)$ has a <u>jump discontinuity</u> at a point $x=x_0$ if the limit from the left $f(x_0^-)$ and the limit from the right $f(x_0^+)$ both exist but not equal

Each function in the Fourier series is periodic with period $2L$. Thus the Fourier series of $f(x)$ on the interval $-L \le x \le L$ is periodic with period $2L$

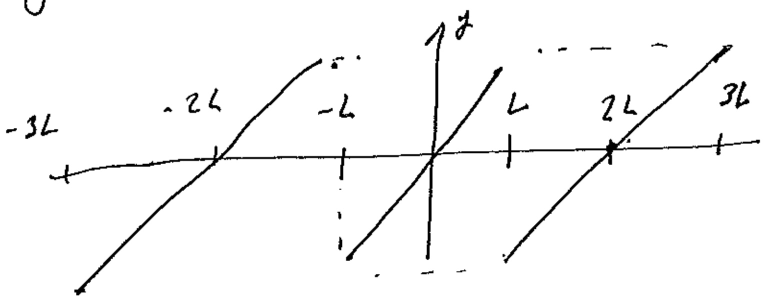
iii) The function does not need to be periodic. We need the <u>periodic extension</u> of $f(x)$. $(x \in [0,L])$

To sketch the periodic extension of $f(x)$, simply sketch $f(x)$ for $-L \le x \le L$ and continually repeat the same pattern with period $2L$ by translating the original sketch for $-L \le x \le L$.

Ex: $f(x) = x \quad 0 < x < L$



periodic extension of $f(x)$



periodic extension of $f(x)$.

Definition of Fourier coefficients and a Fourier Series

We have to distinguish the function $f(x)$ and its Fourier series over $-L \leq x \leq L$

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum b_n \sin\frac{n\pi x}{L} \qquad (*)$$

- This infinite series may not even convergence
- If it converges it may not converge to $f(x)$.

If the series converges

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \qquad n=1,2,\ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \qquad n=1,2,\ldots$$

"Fourier coefficients" of $f(x)$.

Hence Fourier series of $f(x)$ over the interval $-L \leq x \leq L$ is defined to be the series given in $(*)$ where the Fourier coefficients are given above

We immediately note that a Fourier series does not exist unless for example a_0 exists.

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

i.e

$$\left| \int_{-L}^{L} f(x)\,dx \right| \leq \int_{-L}^{L} |f(x)|\,dx < \infty$$

This eliminates certain functions from our considerations. Examples $\frac{1}{x^2}$, $\ln x$, ...

Even if $\int_{-L}^{L} f(x)\,dx$ exists, the infinite series may not converge; furthermore if it converges it may not converge to $f(x)$. We use the notation

$$f(x) \sim a_0 + \sum_{n=1} a_n \cos\frac{n\pi x}{L} + \sum_{n=1} b_n \sin\frac{n\pi x}{L}$$

where $\sim$ means that $f(x)$ is on the left-hand side and the Fourier series of $f(x)$ (on the interval $-L \leq x \leq L$) is on the right hand side (even if the series diverges), but the two functions may be completely different.

Convergence theorem for Fourier series (Fourier's theorem)

If $f(x)$ is piecewise smooth on the interval $-L \le x \le L$, then the Fourier series of $f(x)$ converges

1. to the periodic extention of $f(x)$, where the periodic extension is continuous.

2. to the average of two limits, usually

$$\frac{1}{2}\left[f(x^+)+f(x^-)\right]$$

where the periodic extention has a jump discontinuity

Thus, if $f(x)$ is piecewise smooth, then for $-L < x < L$ (excluding the endpoints)

$$\frac{f(x^+)+f(x^-)}{2} = a_0 + \sum_{n=1} a_n \cos\frac{n\pi x}{L} + \sum_{n=1} b_n \sin\frac{n\pi x}{L}$$

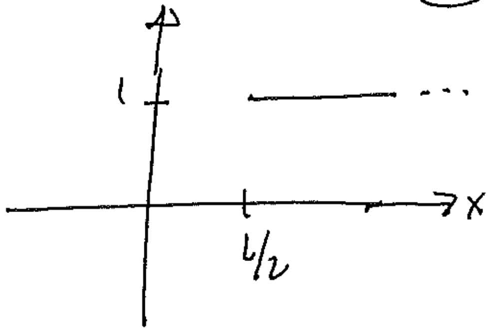
at points where $f(x)$ is continuous, $f(x^+)=f(x^-)$ hence for $-L < x < L$

$$f(x) = a_0 + \sum_{n=1} a_n \cos\frac{n\pi x}{L} + \sum b_n \sin\frac{n\pi x}{L}$$

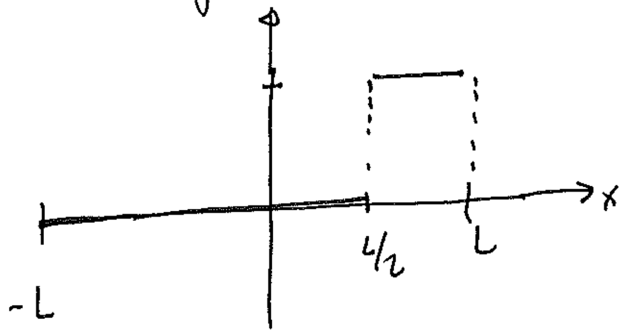
at the end points, $x=L$, or $x=-L$ the infinite series converges to the average of the two values of the periodic extention $\frac{1}{2}\left(f(L)+f(-L)\right)$

Example. consider

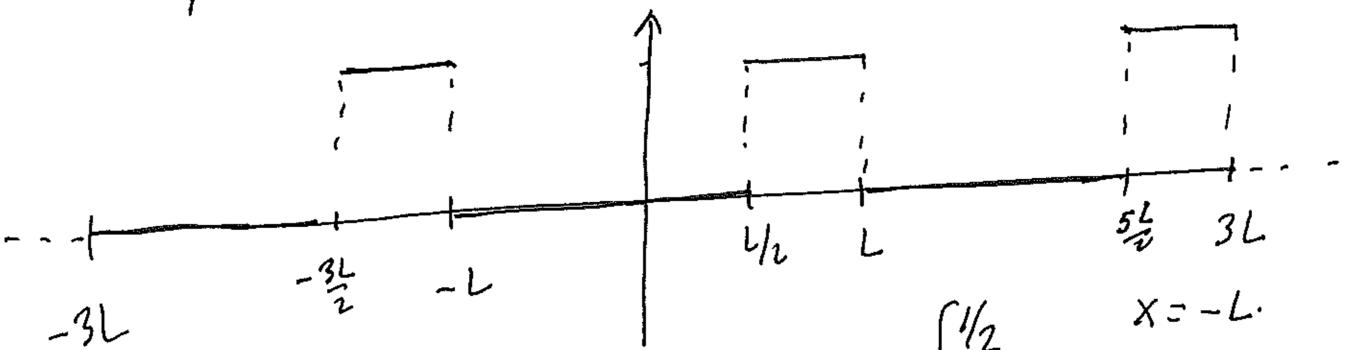
$$f(x) = \begin{cases} 0 & x < L/2 \\ 1 & x > \frac{L}{2} \end{cases}$$



Fourier series of $f(x)$ on $-L \le x \le L$



periodic extension of $f(x)$



$$a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1} b_n \sin\frac{n\pi x}{L} = \begin{cases} 1/2 & x = -L. \\ 0 & -L < x < L/2 \\ 1/2 & x = L/2 \\ 1 & \frac{L}{2} < x < L \\ 1/2 & x = L. \end{cases}$$

Fourier coefficients

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{L/2}^{L} dx = \frac{1}{2L}\cdot(L/2)$$

$$= \frac{1}{4}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{L/2}^{L}\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{L}\,\frac{L}{n\pi}\,\sin\left(\frac{n\pi x}{L}\right)\Big|_{L/2}^{L} = \frac{1}{n\pi}\left(\cancelto{0}{\sin n\pi} - \sin\frac{n\pi}{2}\right)$$

$$= -\frac{1}{n\pi}(-1)^{n+1} = \frac{1}{n\pi}(-1)^{n} \qquad n = \text{odd}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{L/2}^{L}\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{L}\,\frac{L}{n\pi}\left[-\cos\frac{n\pi x}{L}\Big|_{L/2}^{L}\right] = -\frac{1}{n\pi}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)$$

$$= \cancel{\frac{1}{n\pi}(-1)^n} = \frac{1}{n\pi} \qquad n = \text{even}$$

$$= \frac{0}{n\pi} \qquad n = \text{odd}$$

$$\cancel{\frac{1}{4} + \frac{1}{\pi}\sum_{n=1}\frac{1}{n}\sin\frac{n\pi x}{L} + \frac{1}{\pi}}$$

$n = \text{even} = 2k$

$\cos 2k\pi - \cos k\pi$

$= 2 \quad k = \text{odd}$

$= 0 \quad k = \text{even}$

$$f \sim \frac{1}{4} + \frac{1}{\pi}\sum_{n=\text{od}}' \frac{(-1)^n}{n}\cos\frac{n\pi x}{L} + \frac{1}{\pi}\sum_{n=\cancel{\text{even}}\ \text{odd}}' \frac{\sin n\pi x/L}{n.} + \frac{1}{\pi}\sum$$

$$\sim \frac{1}{4} + \frac{1}{\pi}\sum_{n=\text{od.}}' \frac{(-1)^n}{n}\cos\frac{n\pi x}{L} + \frac{1}{\pi}\sum_{n=\text{ev.}} \frac{1}{n}\sin\frac{n\pi x}{L.}$$

$x=0$ $\quad f=0$

$$\frac{1}{4} + \frac{1}{\pi}\sum \frac{(-1)^n}{n} = 0$$

$$\sum_{n=1}^{\infty}{}' \frac{(-1)^n}{n} = -\frac{\pi}{4.} \Rightarrow \sum \frac{(-1)^{n+1}}{n} = \pi/4 \checkmark$$

- For piecewise smooth function $f(x)$, the Fourier series of $f(x)$ is continuous and converges to $f(x)$ for $-L \le x \le L$ if and only if $f(x)$ is continuous and $f(-L) = f(L)$

The condition $f(-L) = f(L)$ insists that the repeated pattern (periodic extension) (with period $2L$) will be continuous at the end points.

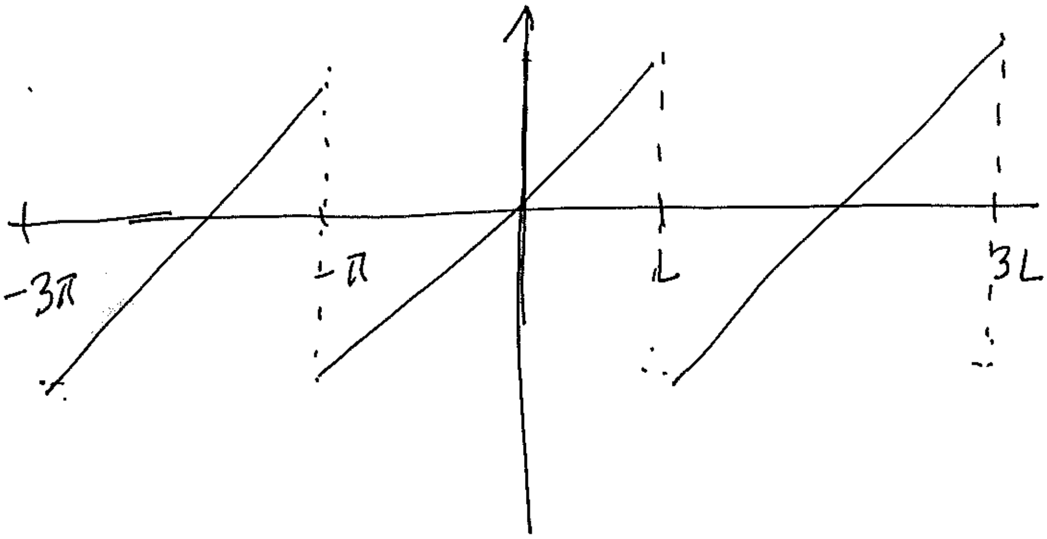
① For piecewise smooth $f(x)$, the Fourier cosine series of $f(x)$ is continuous and converge to $f(x)$ for $0 \le x \le L$ if and only if $f(x)$ is continuous.

② For piecewise smooth functions $f(x)$, the Fourier sine series of $f(x)$ is continuous and converge to $f(x)$ for $0 \le x \le L$ if and only if $f(x)$ is continuous and both $f(0) = 0$ and $f(L) = 0$

3.4
Term-by-term differentiation of Fourier series.

Consider the Fourier sine series of x
In the interval $0 \le x \le L$

$$x = 2\sum_{n=1}^{\infty} \frac{L}{n\pi}(-1)^{n+1} \sin\frac{n\pi x}{L}, \qquad 0 \le x < L$$



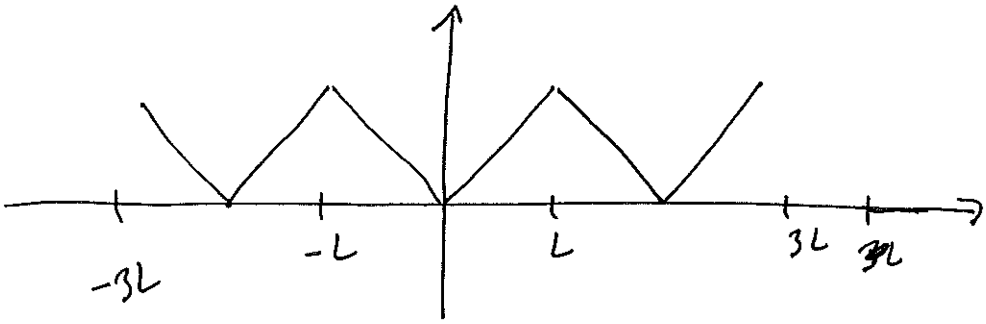
Fourier sine series of x

If we differentiate LHS we get 1.
and differentiate the RHS we get

$$2\sum_{n=1}^{\infty} (-1)^{n+1} \cos\frac{n\pi x}{L}$$

This is a cosine series which is not convergent.
hence it is not equal to 1. (The Fourier cosine series of 1 is 1).

Example : Consider the Fourier cosine series of x



$$X = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n \text{ odd}}' \frac{1}{n^2} \cos \frac{n\pi x}{L} \qquad 0 \le x \le L$$

$x = 0$

$$\sum_{n = \text{odd}} \frac{1}{n^2} = \frac{\pi^2}{8}$$

$$\sum_{n=1}' \frac{1}{n^2} = \frac{\pi^2}{6} \quad \Rightarrow \quad \sum_{n=od}' \frac{1}{n^2} + \sum_{n=ev} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\sum_{n=ev}' \frac{1}{n^2} = \sum_{k=1} \frac{1}{4k^2} = \frac{1}{4} \sum \frac{1}{k^2}$$

2. 4 6

1 2 3

$$\Rightarrow \quad \sum_{n=od}' \frac{1}{n^2} + \frac{1}{4} \sum \frac{1}{n^2} = \sum \frac{1}{n^2}$$

$$\sum_{n=od} \frac{1}{n^2} = \frac{3}{4} \sum_{n=1} \frac{1}{n^2} = \frac{3}{4} \frac{\pi^2}{6} =$$

$$= \pi^2/8$$

⇒ <u>Theorem 1.</u> A Fourier series that is continuous can be differentiated term by term if $f'(x)$ is piecewise smooth.

or

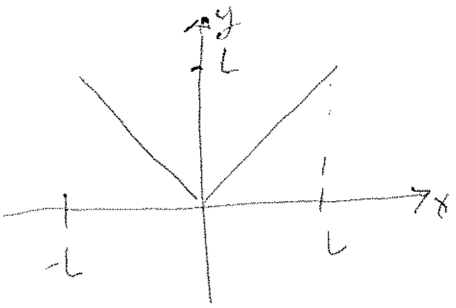
<u>Theorem 2.</u> If $f(x)$ is piecewise smooth, then the Fourier series of a continuous function $f(x)$ can be differentiated term by term if $f(-L) = f(L)$.

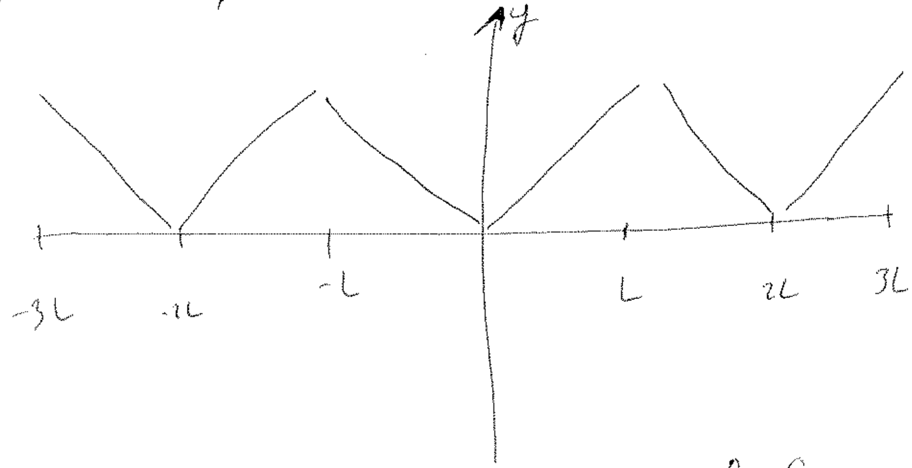
① If $f'(x)$ is piecewise smooth, then a continuous Fourier cosine series of $f(x)$ can be differentiated term by term

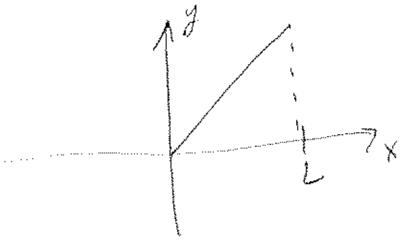
⇒ ② If $f'(x)$ is piecewise smooth, then the Fourier cosine series of a continuous function $f(x)$ can be differentiated term by term.

⇒ ③ If $f'(x)$ is piecewise smooth, then the Fourier series of a continuous function $f(x)$ ~~can~~ only be differentiated term by term if $f(0) = f(L) = 0$

Example: let $f(x) = |x|$, $-L \le x \le L$.







periodic extention of $f(x)$

FS: since $f(x)$ is even it is enough to use cosine series

$$A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L} x\right)$$

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_0^L x\,dx = \frac{L}{2}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx = \frac{2L}{(n\pi)^2}\left[\cos(n\pi) - 1\right]$$

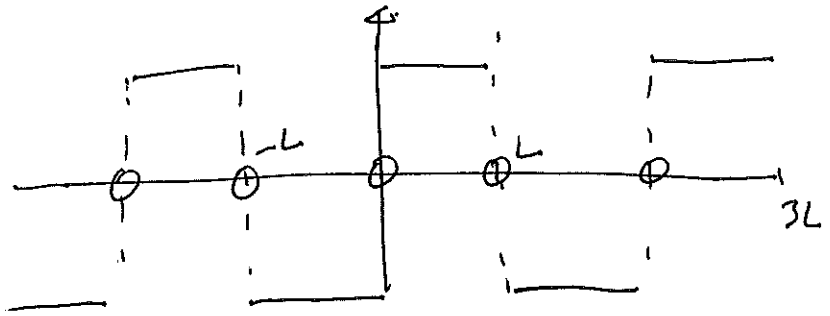
$$= -\frac{4L}{n^2\pi^2} \qquad n \text{ odd}$$

$$x = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{n=1,3,\dots}\frac{1}{n^2}\cos\left(\frac{n\pi}{L}x\right)$$

$$= \frac{L}{2} - \frac{4L}{\pi^2}\left(\cos\frac{\pi}{L}x + \frac{1}{9}\cos\frac{3\pi}{L}x + \frac{1}{16}\cos\frac{5\pi}{L}x + \dots\right)$$

term by term differentiation

$$1 \sim \frac{4}{\pi} \sum_{n=\text{odd}} \frac{1}{n} \sin \frac{n\pi x}{L}, \qquad 0 < x < L.$$



Hence in general

$$f(x) = \sum A_n \cos \frac{n\pi x}{L} \qquad 0 \le x \le L$$

$$f'(x) \sim -\sum \frac{n\pi}{L} A_n \sin \frac{n\pi x}{L}$$

~~Fourier series of f'(x) is continuous and ~ means equality.~~

Where ~ means equality where the FS Fourier sine series of $f'(x)$ is continuous and means the series converges to the average if the Fourier sine series of $f'(x)$ is discontinuous.

Theorem. Let a function $f(x)$ and its derivatives be continuous for $-L \le x \le L$, and let it satisfy the periodicity condition

$$f(-L) = f(L)$$

Then the Fourier series of f converges uniformly to $f(x)$ in the interval $[-L, L]$

proof:

derivative $f'(x)$ is cont. in $-L \le x \le L$

$$f'(x) = \bar{A}_0 + \sum_{n=1} \bar{A}_n \cos\frac{n\pi x}{L} + \sum \bar{B}_n \sin\frac{n\pi x}{L}$$

where

$$\bar{A}_0 = \frac{1}{2L}\int_{-L}^{L} f'(x)\,dx = \frac{1}{2L}\left(f(L) - f(-L)\right) = 0$$

$$\bar{A}_n = \frac{1}{L}\int_{-L}^{L} f'(x)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{n\pi} f'(x)\sin\frac{n\pi x}{L}\Big|_{-L}^{L} - \frac{1}{n\pi}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$\cancel{A_n = \frac{1}{n\pi}}$

$$\bar{A}_n = \frac{1}{L} f(x)\cos\frac{n\pi x}{L}\Big|_{-L}^{L} + \frac{n\pi}{L^2}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{n\pi}{L^2} B_n$$

$$\bar{B}_n = \frac{1}{L}\int_{-L}^{L} f'(x) \sin\frac{n\pi x}{L}\, dx$$

$$= -\frac{n\pi}{L^2}\int_{-L}^{L} f(x) \cos\frac{n\pi x}{L}\, dx$$

$$= -\frac{n\pi}{L^2} A_n$$

$$f(x) \sim A_0 + \sum A_n \cos\frac{n\pi x}{L} + \sum B_n \sin\frac{n\pi x}{L}$$

$$\left| A_0 + \sum A_n \cos\frac{n\pi x}{L} + \sum B_n \sin\frac{n\pi x}{L} \right|^2$$

$$< \left| |A_0|^2 + \sum |A_n| + \sum B_n \right|$$

$$< |A_0| + \frac{L^2}{\pi} \sum \frac{|\bar{B}_n|}{n} + \frac{L^2}{\pi} \sum \frac{|\bar{A}_n|}{n}$$

$$\left(\sum \frac{|\bar{B}_n|}{n} \right)^2 \leq \sum |\bar{B}_n|^2 \sum \frac{1}{n^2} < \infty$$

$$\Rightarrow \left| \qquad \right| \leq |A_0| + \frac{L^2}{\pi^2} \left(\sum |A_n \right.$$

continuity of f' can be omitted. $f(x)$ is a piecewise smoth function in $[-L, L]$ and $f(L) = f(-L)$.

$f'(x)$ is piecewise smooth (finite number of jumps)
$f(x)$ has only discontinuities of the first kind ($f(x^+) \neq f(x^-)$). $\Rightarrow$

$$\lim_{n \to \infty} \left[A_0 + \sum A_n \cos\frac{n\pi x}{L} + \sum B_n \sin\frac{n\pi x}{L} \right]$$

$$= \begin{cases} \frac{1}{2}[f(x^+) + f(x^-)] & -L < x < L \\ \frac{1}{2}[f(L) + f(-L)] & x = \pm L \end{cases}$$